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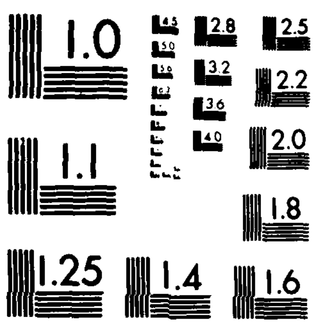

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SOME NEW GRAVITY WAVES IN WATER OF FINITE DEPTH

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UNIVERSITY OF WISCONSIN - MADISON
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Jean-Marc Vanden-Broeck*

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ABSTRACT

The results of Chen and Saffman, showing that periodic gravity waves in water of infinite depth are not unique, are generalized for waves in water of finite depth. Some new types of waves are discovered and discussed.

AMS (MOS) Subject Classification: 76B15

Key Words: gravity waves, bifurcation

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SIGNIFICANCE AND EXPLANATION

In a recent paper Chen and Saffman [1] showed that periodic gravity waves in water of infinite depth are not unique. They presented explicit computations for new families of waves which they termed irregular waves.

In the present paper ~~we~~ the authors give conclusive numerical evidence that periodic gravity waves in water of arbitrary uniform depth are not unique. Explicit computations of irregular waves in water of finite depth are presented.

In addition, ~~we~~ they show that Chen and Saffman's [1] bifurcation point for an irregular wave of class 2 is not unique. Our results suggest the existence of an infinite number of such bifurcation points.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SOME NEW GRAVITY WAVES IN WATER
OF FINITE DEPTH

Jean-Marc Vanden-Broeck*

I. Introduction

In a recent paper Chen and Saffman [1] gave conclusive numerical evidence that gravity waves in water of infinite depth are not unique. They showed that the classical Stokes' waves can bifurcate at large amplitude into new families of waves which they termed irregular waves. They labelled the irregular waves with a "class number". The class number gives the number of crests per wavelength.

Chen and Saffman [1] presented detailed computations for irregular waves of class 2 and 3. Further results dealing with irregular waves in water of infinite depth have been obtained by Saffman [2] and Olfe and Rottman [3].

In this paper we present explicit computations for irregular waves of class 2 in water of finite depth. We show that Stokes' waves in water of arbitrary uniform depth can bifurcate at large amplitude into irregular waves of class 2. The branches emanating from the bifurcation points are computed for various values of the depth.

Chen and Saffman [1] found that Stokes' waves in water of infinite depth bifurcate at $\frac{h}{\lambda} \sim 0.13$ into irregular waves of class 2. Here h is the wave height and λ is the wave length.

We show that this bifurcation point is not unique. We found explicitly another bifurcation point at $\frac{h}{\lambda} \sim 0.140$. Our results suggest the existence of an infinite number of such bifurcation points.

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As the depth tends to zero the classical Stokes' wave and the irregular waves approach the same solitary wave configuration. Therefore the non-uniqueness of periodic gravity waves does not imply the non-uniqueness of solitary waves.

The problem is formulated in section 2. The numerical procedure is outlined in section 3 and the results are discussed in section 4.

II. Formulation

We consider two-dimensional periodic waves of wavelength λ and phase velocity c propagating under the influence of gravity g over a horizontal bottom. We choose a frame of reference in which the waves are steady as is the fluid motion which is assumed to be potential. The x-axis is parallel to the bottom and the y-axis is a line of symmetry of the wave. We make the variables dimensionless by referring them to the velocity c and the length λ.

We choose the complex potential $f = \phi + i\psi$ as the independent variable. Let the stream function ψ assume the values 0 and $-Q$ on the free-surface and on the bottom respectively. We define the undisturbed fluid depth by

$$d = \frac{Q}{c} \quad . \tag{1}$$

We denote by $x(\phi)$ and $y(\phi)$ the values of x and y on the free-surface $\psi = 0$. Following Vanden-Broeck and Schwartz [4] we derive the following integro-differential relations between $x'(\phi)$ and $y'(\phi)$:

$$\begin{aligned} x'(\phi) - 1 = &-\int_0^{1/2} y'(s)[\operatorname{cotg} \pi(s-\phi) + \operatorname{cotg} \pi(s+\phi)]ds \\ &+ 2r_0^2 \int_0^{1/2} \frac{[x'(s)-1][r_0^2-\cos 2\pi(s-\phi)] - y'(s)\sin 2\pi(s-\phi)}{1 + r_0^4 - 2r_0^2 \cos 2\pi(s-\phi)} ds \\ &+ 2r_0^2 \int_0^{1/2} \frac{[x'(s)-1][r_0^2- \cos 2\pi(s+\phi)] - y'(s)\sin 2\pi(s+\phi)}{1 + r_0^4 - 2r_0^2 \cos 2\pi(s+\phi)} ds \end{aligned} \tag{2}$$

$$y(\phi) + \frac{\mu}{4\pi} \{\frac{1}{[x'(\phi)]^2 + [y'(\phi)]^2} - 1\} = 0 \quad . \tag{3}$$

Here μ and r_0 are defined by

$$\mu = 2\pi c^2/g\lambda \tag{4}$$

$$r_0 = \exp[-2\pi d/\lambda] \quad . \tag{5}$$

The choice of the Bernoulli constant in (3) fixes the origin of y as the level of the free surface at which the velocity is equal to $\mu^{1/2}$.

In addition to the parameters r_0 and μ, a wave is characterized by a third parameter which is a measure of the wave steepness. We choose this parameter to be

$$\varepsilon = \frac{4\pi y_c}{\mu} \tag{6}$$

where y_c is the elevation of the crests of the wave. For the highest wave, the velocity at the crest vanishes. Thus from (3), $y_c = \frac{\mu}{4\pi}$ so $\varepsilon = 1$ for the highest wave. In general, ε ranges between 0 and 1.

For given values of r_0 and ε (2) and (3) define a nonlinear integro-differential equation for the unknown functions $x(\phi)$, $y(\phi)$ and the constant μ. Vanden-Broeck and Schwartz [4] derived an efficient numerical procedure to solve this equation. In the next section we use their procedure to compute irregular waves of class 2.

III. Numerical procedure

Following Chen and Saffman [1] we define a new variable β by the transformation

$$\phi = 2\pi\beta - \frac{\alpha}{n} \sin 2\pi n\beta \quad . \tag{7}$$

Here n is the number of crests per wavelength. Thus we choose $n = 1$ for classical Stokes' waves and $n = 2$ for irregular waves of class 2. Next we define the mesh points

$$\beta_I = [(I-1)/N]\pi \qquad I = 1,\ldots,N+1 \quad . \tag{8}$$

The parameter α in (7) is used to concentrate mesh points near the crest. The closest α is to one, the greater the concentration. For steep waves we chose $\alpha = 0.999$.

Following Vanden-Broeck and Schwarts [4] we discretize (2) and (3). For given values of r_0 and ε we obtain a system of N nonlinear algebraic equations for the N unknowns $y_{\beta_I} = \left(\frac{\partial y}{\partial \beta}\right)_{\beta=\beta_I}$ $I = 2,\ldots,N$ and μ. This system may be written symbolically as

$$F_I(y_{\beta_2},\ldots,y_{\beta_N}, \mu) = 0 \qquad I = 1,\ldots,N \quad . \tag{9}$$

Vanden-Broeck and Schwartz [4] solved the system (9) by Newton's iterations. They chose $n = 1$ in (7) and obtain accurate solutions for classical Stokes' waves. In order to find bifurcation points for irregular waves of class 2, we repeated Vanden-Broeck and Schwartz's calculations with $n = 2$ in (7). No significant differences were found between the results with $n = 1$ and those with $n = 2$.

The bifurcation points were found by monitoring the sign of the Jacobian of the system (9). The new branches emanating from these bifurcation points were computed by using Keller's [5] method. The details of our numerical procedure follow closely the scheme described by Chen and Saffman [1]. Therefore, they will not be repeated here.

IV. Discussion of the results

The procedures outline in section 3 was used to compute irregular waves of class 2 for various values of r_0 and ε.

In Figures 1 and 2 we present typical profiles of irregular waves of class 2 for $r_0^2 = 0.4$ and $r_0^2 = 0.8$. For r_0 small, the profiles are very similar to those presented by Chen and Saffman [1]. As r_0 increases the wave develops narrow crests and broad troughs. This behavior appears clearly in the profiles of Figures 1 and 2.

In Figures 3 - 5 we show graphs of $\tau = (\mu-\mu_0)/\mu_0$ against ε. Here μ_0 is the value of μ for infinitesimal waves, i.e.

$$\mu_0 = \frac{1-r_0^4}{2(1+r_0^4)} \quad . \tag{10}$$

The solid curves correspond to the classical Stokes' waves with $n = 2$. Chen and Saffman [1] termed these waves "regular waves of class 2". The dashed curves correspond to the irregular waves of class 2.

For water of infinite depth, i.e. $r_0 = 0$, the regular waves of class 2 can bifurcate into an irregular wave of class 2 at $\varepsilon \sim 0.88$. This is the bifurcation point discovered by Chen and Saffman [1]. The branch emanating from that point is in good agreement with the numerical results of Chen and Saffman [1].

We found that Chen and Saffman's bifurcation point is not unique. We discovered another bifurcation at $\varepsilon \sim 0.993$. The new branch emanating from that point is shown in Figure 3.

As the depth tends to zero, i.e. as $r_0 \to 1$, the value of ε corresponding to Chen and Saffman's bifurcation point approaches the value of ε for which τ is maximum (see Figure 3 - 5). Furthermore the distance between the dashed curves (irregular waves of class 2) and the solid curves

(regular waves of class 2) tends to zero as $r_0 \to 1$. Therefore the non-uniqueness of periodic gravity waves does not imply the non-uniqueness of solitary waves.

This result can be interpreted in the following way. Solitary waves are the limit of periodic waves as $\frac{\lambda}{H} \to \infty$. Therefore a solitary provides a good approximation for a periodic wave with $\frac{\lambda}{H} = L$ where $L >> 1$. The existence of a value of ε for which τ is maximum implies that two solitary waves of different amplitude ε can travel at the same speed τ. Two such waves can be used to approximate an irregular wave of class 2 with $\frac{\lambda}{H} = 2L$. This explains why the dashed curves and the solid curves of Figure 2 coincide in the limit as $r_0 \to 1$.

Longuet-Higgins and Fox [6] showed that μ oscillates infinitely often as $\varepsilon \to 1$. As $r_0 \to 1$, our new bifurcation point approaches the value of ε corresponding to the first minimum as τ.

We were not able to compute waves past the first minimum of τ. However it seems likely that a third bifurcation point exists between the first minimum and the second maximum of τ. As $r_0 \to 1$, this bifurcation point should approach the value of ε corresponding to the second maximum of τ.

We suggest that an infinite number of such bifurcation points exist. As $r_0 \to 1$, these bifurcation points approach the values of ε corresponding to the local minima and maxima of τ.

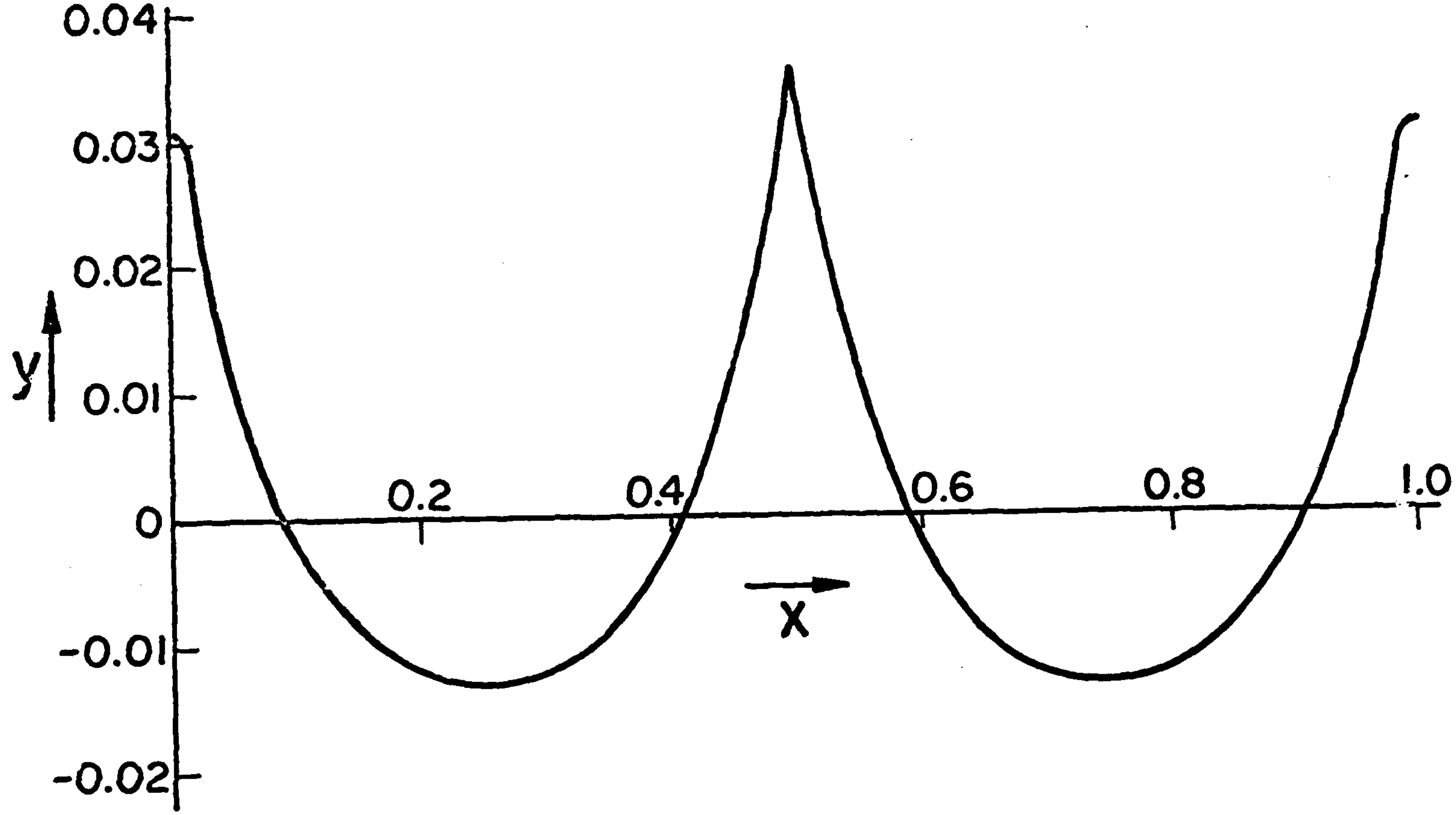

Figure 1:

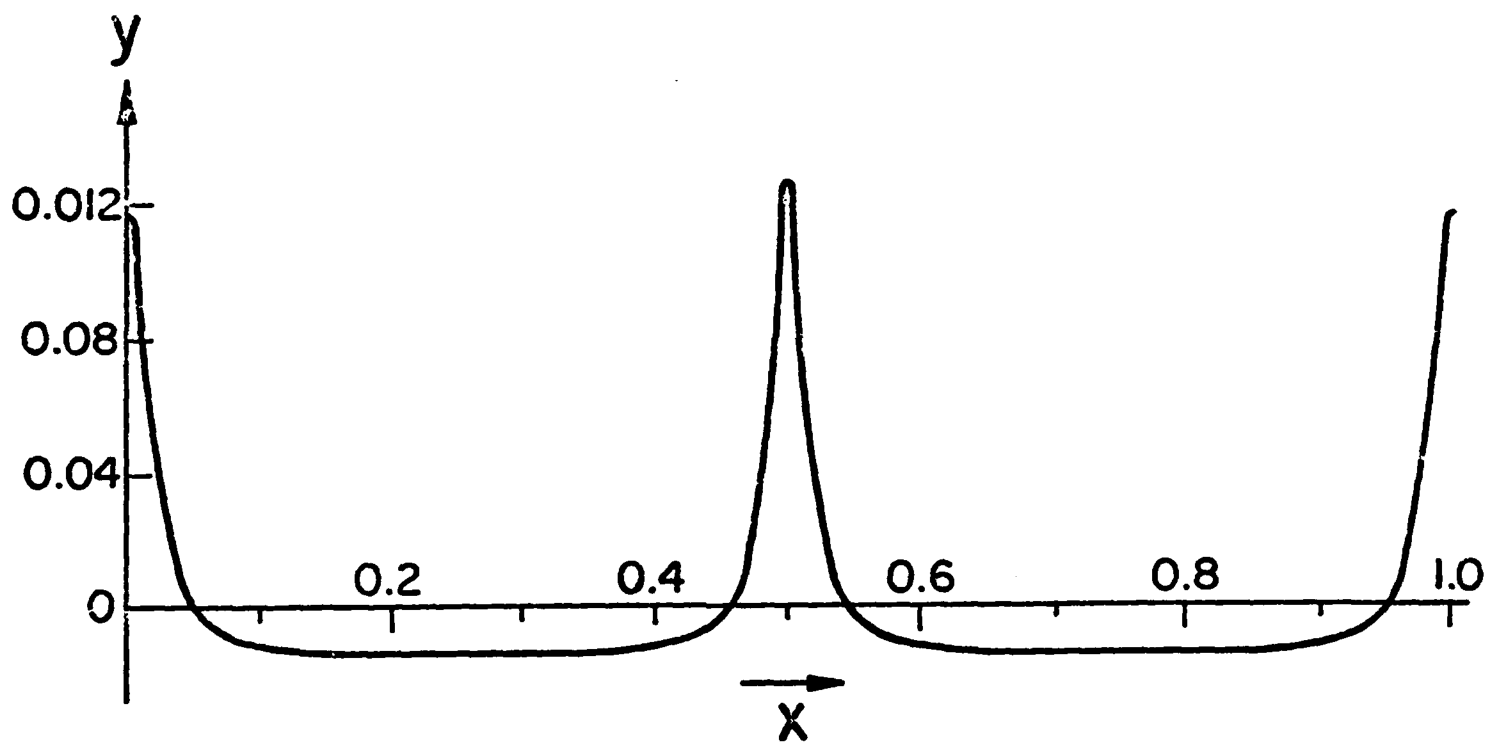

Figure 2:

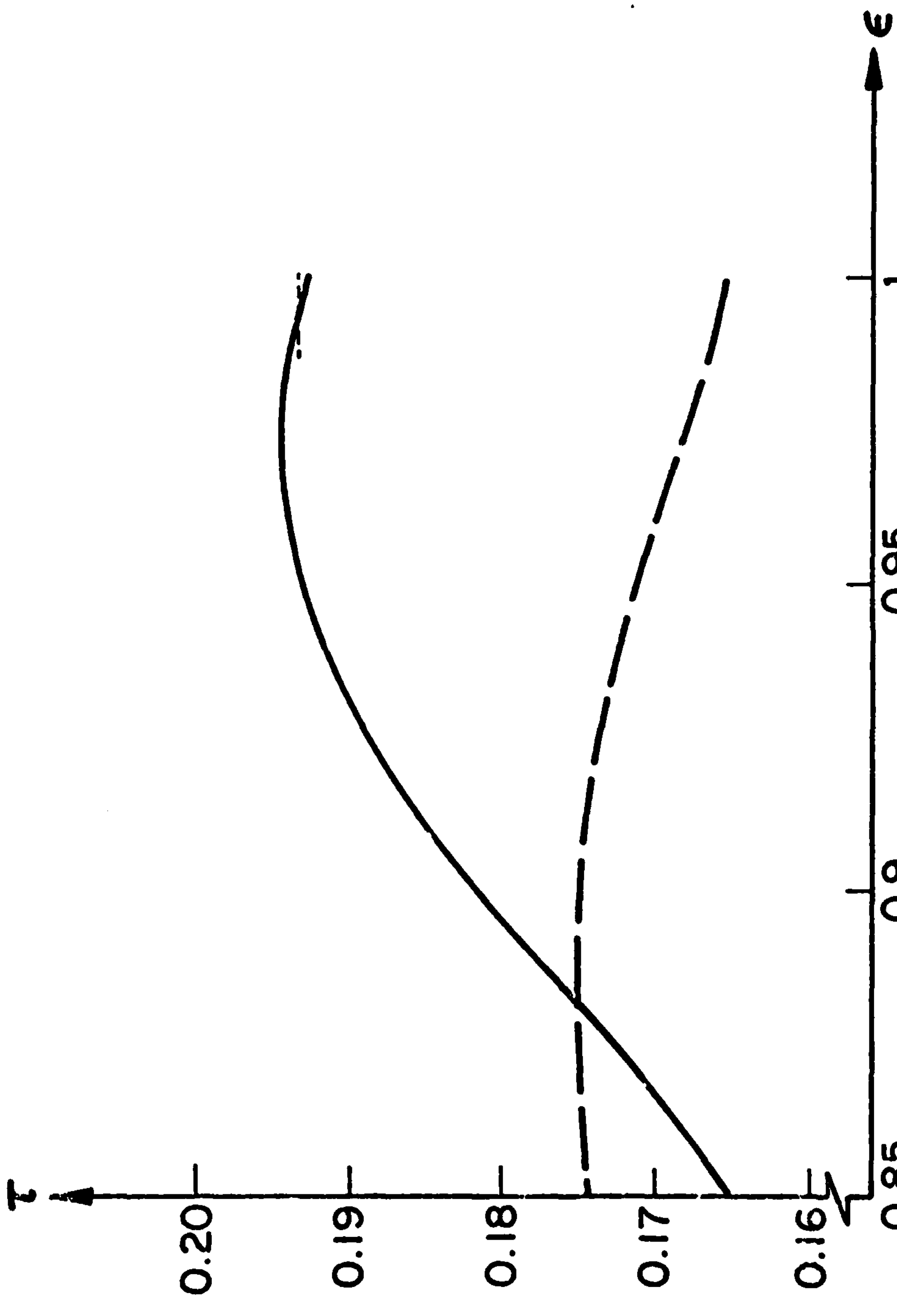

Figure 3:

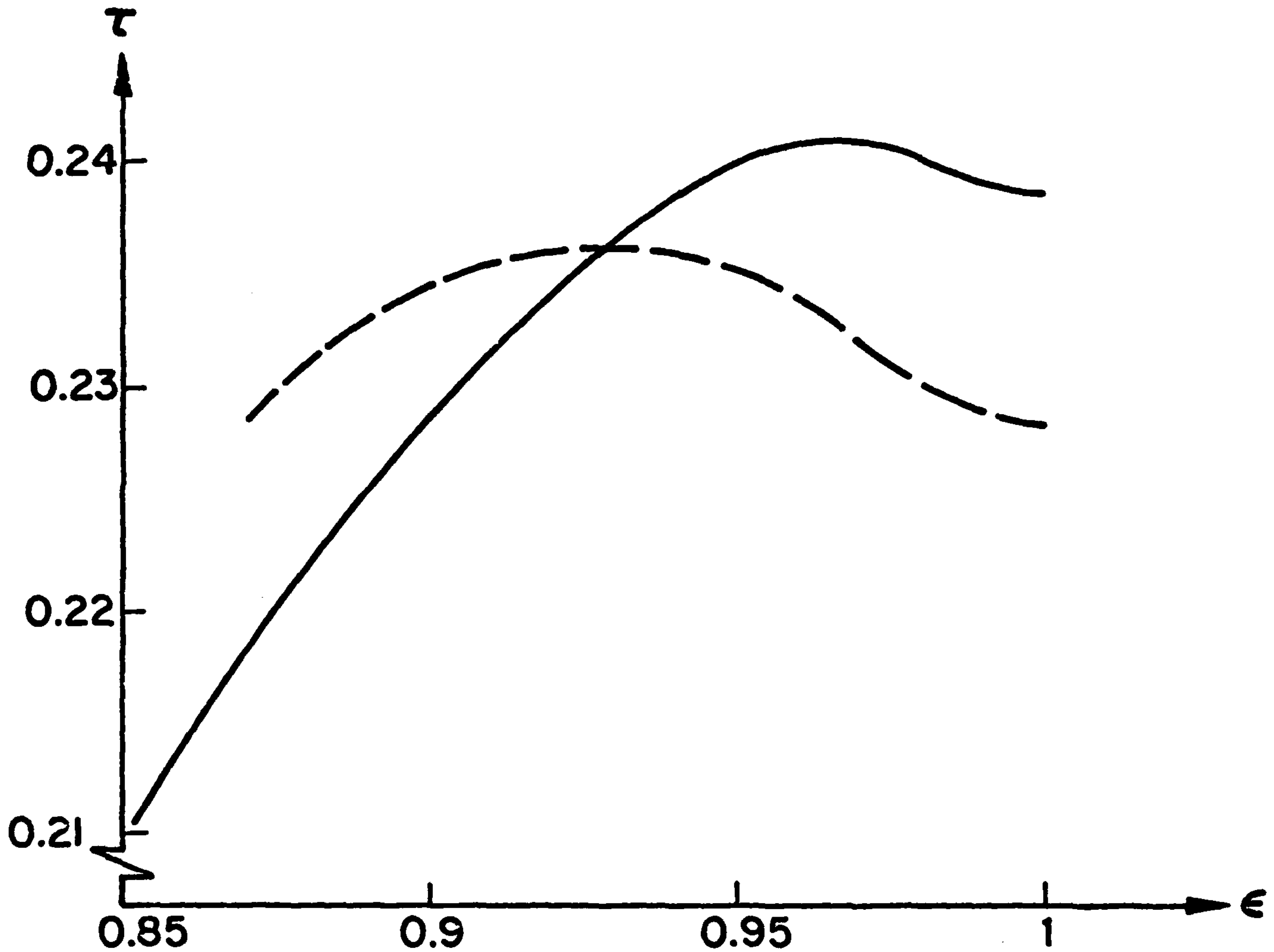

Figure 4:

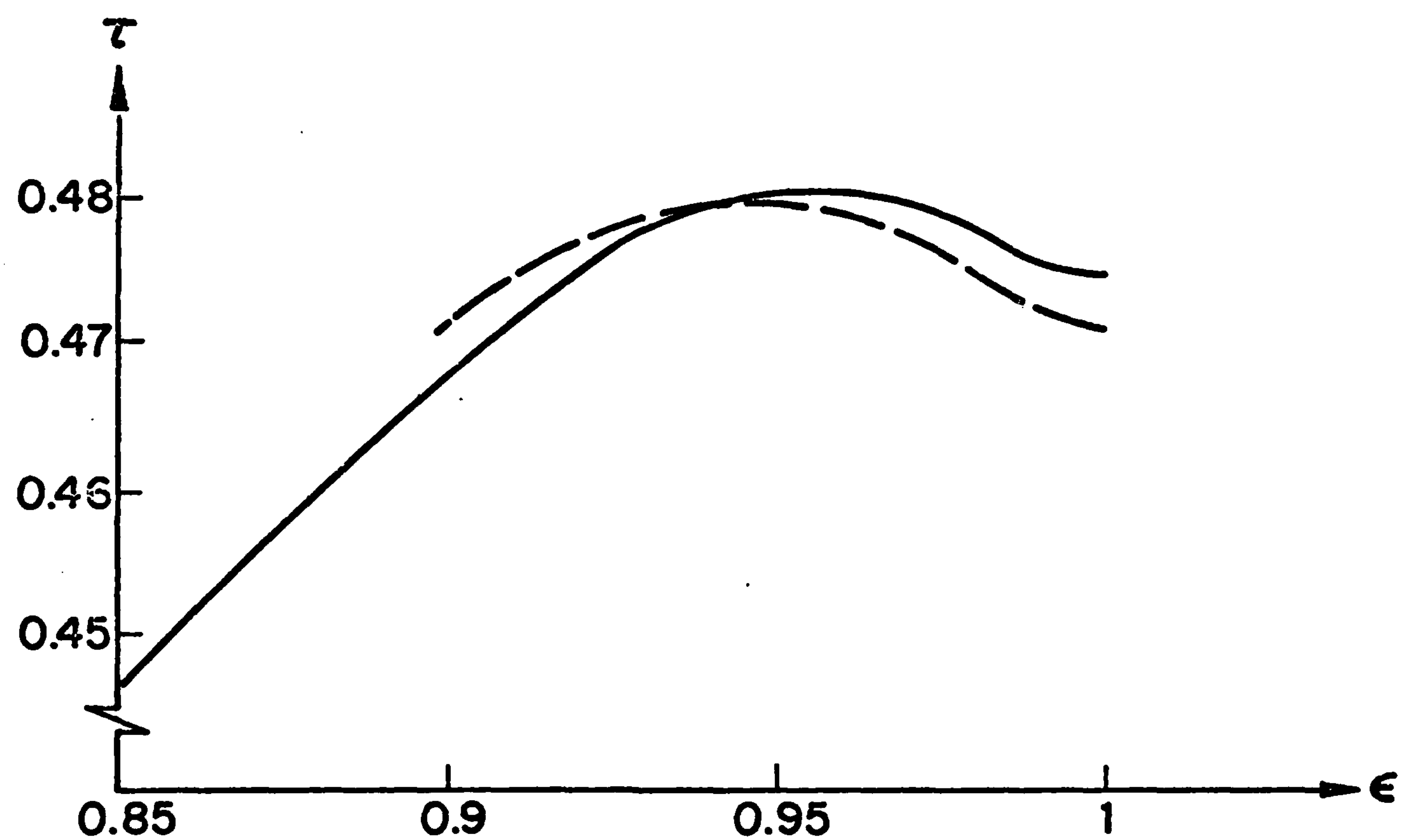

Figure 5:

Captions for Figures

Figure 1. Free surface profile of an irregular wave of class 2 for $r_0^2 = 0.4$ and $\varepsilon = 0.997$.

Figure 2. Same as Figure 1 with $r_0^2 = 0.8$ and $\varepsilon = 0.984$.

Figure 3. Values of τ versus ε for $r_0^2 = 0$. The solid curve corresponds to regular waves of class 2 and the dashed curves to irregular waves of class 2.

Figure 4. Same as Figure 3 with $r_0^2 = 0.4$.

Figure 5. Same as Figure 3 with $r_0^2 = 0.8$.

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